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1993 J. Phys. A: Math. Gen. 26 4451

(http://iopscience.iop.org/0305-4470/26/17/049)

J. Phys. A: Math. Gen. 26 (1993) 4451–4462. Printed in the UK

Bound states of a fermion and a Dirac dyon in Robertson–Walker metric

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Received 22 September 1992, in final form 13 April 1993

Abstract. The Dirac equation for a system of fermion and dyon is separated into decoupled ordinary differential equation in the Robertson–Walker metric. Energy levels of bound states for a fermion and a dyon with charge $Z_d < Z_d^c$ and for $j \geq |q| + \frac{1}{2}$ are obtained. If dyons are indeed present in the universe, they have an interesting astronomical observation.

1. Introduction

Since the most satisfactory way of writing Dirac equation is in the framework of the spinor formalism, the spinorial basis of the Newman–Penrose formalism has been the subject of interest and discussion for a long time [1]. Chandrasekhar [2] has separated the Dirac equation in the Kerr geometry. The Dirac equation for an electron around a Kerr–Newman black hole has also been separated into decoupled ordinary differential equations [3].

On the other hand, the problem of bound states of a fermion in a fixed Dirac monopole [4] or in a non-Abelian monopole [5] has been extensively discussed [6–27]. As is well known for the system of a fermion and a Dirac monopole, there is the Lipkin–Weisberger–Peshkin (LWP) difficulty [26] in the angular momentum states $j = |q| - \frac{1}{2}$ which shows up in the fermion's radial wavefunctions at the origin. The radial wavefunctions of the fermion in the angular momentum state $j = |q| - \frac{1}{2}$ do not vanish at the origin. This means that the fermion in these states goes through the monopole; thus, the Hamiltonian of the system is ill-defined at the origin. To avoid this difficulty, an infinitesimal extra magnetic moment is endowed to the fermion by Kazama and Yang [7, 8]. In [11] the authors showed that for the system of a fermion and a Dirac dyon there is also the LWP difficulty in the angular momentum states $j \geq |q| + \frac{1}{2}$ when the dyon charge Z_d exceeds some critical value Z_d^c. In order to avoid the LWP difficulty, besides the Kazama–Yang term $-(Kq/2Mr^3)\beta\boldsymbol{\Sigma}\cdot\boldsymbol{r}$, the term $\mathrm{i}(KZZ_d e^2/2Mr^3)\boldsymbol{\gamma}\cdot\boldsymbol{r}$ should be considered also. But in the case $Z_d < Z_d^c$, the Hamiltonian of the system is well defined at the origin, so we can solve the bound-state energy for $j \geq |q| + \frac{1}{2}$ without the Kazama–Yang term and the term

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$i(KZZ_d e^2/2Mr^3)\gamma\cdot r$ [11]. The results show that the bound-state energy is hydrogen-like [15] in the flat space-time.

In this paper, the analysis starts with the Dirac equation coupled to general gravitational and electromagnetic fields. The Dirac equation for a system of fermion and dyon is separated into decoupled ordinary differential equation in the Robertson–Walker metric. Energy levels of bound states for a fermion and a Dirac dyon with charge $Z_d < Z_d^c$ and for $j \geqslant |q| + \frac{1}{2}$ are obtained by using the perturbation expansion in the closed or open Robertson–Walker metric. These energy levels coincide with those of the normal solution in the flat space-time when the cosmological radius goes to infinity.

2. The equation of dyon–fermion system in the Robertson–Walker metric

The analysis starts with the Dirac equation coupled to general gravitational and electromagnetic fields. In two-component spinor notation [2], the equation is [3]

$$\sqrt{2}(\nabla_{AB'} + \mathrm{i}ZeA_{AB'})P^A + \mathrm{i}M\bar{Q}_{B'} = 0 \tag{2.1}$$

$$\sqrt{2}(\nabla_{AB'} - \mathrm{i}ZeA_{AB'})Q^A + \mathrm{i}M\bar{P}_{B'} = 0 \tag{2.2}$$

where $\nabla_{AB'}$ is the symbol for covariant differential, $A_{AB'}$ is the electromagnetic vector field potential, Ze is the charge or coupling constant of the fermion to the vector field, M is the particle mass, and P^A and Q^A are the two-component spinors representing the wavefunction. The bar denotes complex conjugation, $\bar{Q}_{B'}$, is the complex conjugate of Q_B. It is convenient to consider the complex conjugate of (2.2) and further write

$$F_1 = P^0 \qquad F_2 = P^1 \qquad G_1 = \bar{Q}^{1'} \qquad \text{and } G_2 = -\bar{Q}^{0'}. \tag{2.3}$$

The resulting equations are

$$\begin{aligned}
&\sqrt{2}(D + \varepsilon - \rho + \mathrm{i}eA_\mu l^\mu)F_1 + \sqrt{2}(\bar{\delta} + \pi - \alpha + \mathrm{i}eA_\mu \bar{m}^\mu)F_2 = \mathrm{i}MG_1 \\
&\sqrt{2}(\Delta + \mu - \gamma + \mathrm{i}eA_\mu n^\mu)F_2 + \sqrt{2}(\delta + \beta - \tau + \mathrm{i}eA_\mu m^\mu)F_1 = \mathrm{i}MG_2 \\
&\sqrt{2}(D + \bar{\varepsilon} - \bar{\rho} + \mathrm{i}eA_\mu l^\mu)G_2 - \sqrt{2}(\delta + \bar{\pi} - \bar{\alpha} + \mathrm{i}eA_\mu m^\mu)G_1 = \mathrm{i}MF_2 \\
&\sqrt{2}(\Delta + \bar{\mu} - \bar{\gamma} + \mathrm{i}eA_\mu n^\mu)G_1 - \sqrt{2}(\bar{\delta} + \bar{\beta} - \bar{\tau} + \mathrm{i}eA_\mu \bar{m}^\mu)G_2 = \mathrm{i}MF_1.
\end{aligned} \tag{2.4}$$

The Robertson-Walker metric is given by

$$\mathrm{d}S^2 = a^2\{\mathrm{d}\tau^2 - [\mathrm{d}\chi^2 + S^2(\chi)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)]\} \tag{2.5}$$

where a is the cosmological radius and $s(\chi) = \sin\chi$, χ and $\sinh\chi$, respectively, to closed, flat and open cosmological models. We will first discuss closed space-time. From (2.5), the following null tetrads $(l_\mu, n_\mu, m_\mu, \bar{m}_\mu)$ can be constructed

$$\begin{aligned}
l_\mu &= \frac{1}{\sqrt{2}}(a, a, 0, 0) \\
n_\mu &= \frac{1}{\sqrt{2}}(a, -a, 0, 0) \\
m_\mu &= \frac{1}{\sqrt{2}}(0, 0, a\sin\chi, \mathrm{i}a\sin\chi\sin\theta) \\
\bar{m}_\mu &= \frac{1}{\sqrt{2}}(0, 0, a\sin\chi, -\mathrm{i}a\sin\chi\sin\theta).
\end{aligned} \tag{2.6}$$

The null vectors satisfy the orthogonality conditions, $l^\mu n_\mu = 1$, $m^\mu \bar{m}_\mu = -1$, while all the remaining scalar products are zero. Thus, the directional derivatives is given by

$$
\begin{aligned}
D &= l^\mu \frac{\partial}{\partial \chi^\mu} = \frac{1}{\sqrt{2}a}\left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \chi}\right) \\
\Delta &= n^\mu \frac{\partial}{\partial \chi^\mu} = \frac{1}{\sqrt{2}a}\left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial \chi}\right) \\
\delta &= m^\mu \frac{\partial}{\partial \chi^\mu} = \frac{1}{\sqrt{2}a}\left(-\varphi\right) \\
\bar{\delta} &= \bar{m}^\mu \frac{\partial}{\partial \chi^\mu} = \frac{1}{\sqrt{2}a}\left(-\frac{1}{\sin\chi}\frac{\partial}{\partial \theta} + \frac{\mathrm{i}}{\sin\chi \sin\theta}\frac{\partial}{\partial \varphi}\right).
\end{aligned}
\tag{2.7}
$$

The non-zero spin coefficients are obtained from (2.6) as follows

$$
\begin{aligned}
\varepsilon &= -\gamma = \frac{1}{2\sqrt{2}a}\frac{\dot{a}}{a} \\
\rho &= -\frac{1}{\sqrt{2}a}\left(\frac{\dot{a}}{a} - \cot\chi\right) \\
\alpha &= -\beta = (2\sqrt{2}a \sin\chi)^{-1}\cot\theta \\
\mu &= \frac{1}{\sqrt{2}a}\left(\frac{\dot{a}}{a} + \cot\chi\right),
\end{aligned}
\tag{2.8}
$$

where $\dot{a} = \mathrm{d}a/\mathrm{d}\tau$.

In order to simplify the calculation, we set the position of the dyon at the polar point. Then the four-component electromagnetic vector field potential A_μ is given by

$$
A_\mu = (A_0, 0, 0, A_\varphi) = \left(\frac{\lambda}{a\chi}, 0, 0, A_\varphi\right) \tag{2.9}
$$

where $\lambda = ZZ_\mathrm{d}e^2$, z is the electric charge of the fermion which is an integer, Z_d is the electric charge of the dyon, which need not be an integer. Here A_0 is different from A_0 given in [28], but to first-order approximation they are the same. A_φ is the vector potential of the dyon. In order to remove the string of singularities, A_φ is defined in terms of two or more functions in a corresponding number of overlapping regions [6]. Substituting (2.7), (2.8) and (2.9) into (2.4), we have

$$\left[\mathrm{i}\gamma^\mu\partial_\mu - \mathrm{i}\gamma^0\frac{3}{2}\frac{\dot{a}}{a} + \mathrm{i}\gamma^1\cot\chi + \mathrm{i}\gamma^2(\tfrac{1}{2}\cot\theta) - \gamma^0\frac{\lambda}{a\chi} - \gamma^3 eA_\varphi - M\right]\Phi = 0 \tag{2.10}$$

where

$$\gamma^0 = \frac{1}{a}\hat{\gamma}^0 \qquad \gamma^1 = \frac{1}{a}\hat{\gamma}^1 \qquad \gamma^2 = \frac{1}{a\sin\chi}\hat{\gamma}^2 \qquad \gamma^3 = \frac{1}{a\sin\chi\sin\theta}\hat{\gamma}^3, \tag{2.11}$$

and

$$\Phi = \begin{pmatrix} G_1 \\ G_2 \\ F_1 \\ F_2 \end{pmatrix} \tag{2.12}$$

where $\hat{\gamma}$ are the chiral representations of normal Dirac matrices in flat space-time

$$\hat{\gamma}^0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \qquad \hat{\gamma}^\mathrm{i} = \begin{pmatrix} 0 & \sigma^\mathrm{i} \\ -\sigma^\mathrm{i} & 0 \end{pmatrix}. \tag{2.13}$$

Relation with the Dirac representation:

$$\gamma^\mu_{\text{chiral}} = U\gamma^\mu_{\text{Dirac}}U^\dagger \tag{2.14}$$

where

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{I} & -\mathrm{I} \\ \mathrm{I} & \mathrm{I} \end{pmatrix}. \tag{2.15}$$

The wavefunctions ψ_{chiral} are four-components spinors defined by

$$\psi_{\text{chiral}} = a^{3/2}\sin\chi(\sin\theta)^{1/2}\,\Phi = \begin{pmatrix} \hat{G}_1 \\ \hat{G}_2 \\ \hat{F}_1 \\ \hat{F}_2 \end{pmatrix}. \tag{2.16}$$

By substituting (2.14) into (2.10), we have

$$[\mathrm{i}\gamma^\mu(\partial_\mu + \mathrm{i}eA_\mu) - M]\psi_{\text{chiral}} = 0. \tag{2.17}$$

Its component form is

$$\begin{aligned}
&\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\chi} + \mathrm{i}\frac{\lambda}{a\chi}\right)\hat{F}_1 - \frac{1}{\sin\chi}\left[\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sin\theta}\left(\frac{\partial}{\partial\theta} + \mathrm{i}eA_\varphi\right)\right]\hat{F}_2 - \mathrm{i}Ma\hat{G}_1 = 0 \\
&\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\chi} + \mathrm{i}\frac{\lambda}{a\chi}\right)\hat{F}_2 - \frac{1}{\sin\chi}\left[\frac{\partial}{\partial\theta} + \frac{\mathrm{i}}{\sin\theta}\left(\frac{\partial}{\partial\theta} + \mathrm{i}eA_\varphi\right)\right]\hat{F}_1 - \mathrm{i}Ma\hat{G}_2 = 0 \\
&\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\chi} + \mathrm{i}\frac{\lambda}{a\chi}\right)\hat{G}_1 - \frac{1}{\sin\chi}\left[\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sin\theta}\left(\frac{\partial}{\partial\theta} + \mathrm{i}eA_\varphi\right)\right]\hat{G}_2 - \mathrm{i}Ma\hat{F}_1 = 0 \\
&\left(\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\chi} + \mathrm{i}\frac{\lambda}{a\chi}\right)\hat{G}_2 - \frac{1}{\sin\chi}\left[\frac{\partial}{\partial\theta} + \frac{\mathrm{i}}{\sin\theta}\left(\frac{\partial}{\partial\theta} + \mathrm{i}eA_\varphi\right)\right]\hat{G}_1 - \mathrm{i}Ma\hat{F}_2 = 0.
\end{aligned} \tag{2.18}$$

It is reasonable to assume that the radius of the universe $a(\tau)$ is not changed in the transition process (less than 10^{-9} s), that is $a(\tau)$ is constant in the whole calculation.

Under the Dirac representation, the wavefunction $\psi = U^{+}\psi_{\text{chiral}}$. For the states of $j \geqslant |q| + \frac{1}{2}$ there are two types of simultaneous eigensections of J^2, J_z and H in analogy to the treatment in the system of monopole-fermion [7]:

$$\text{Type A} \qquad \psi^{(1)}_{jm} = \frac{1}{\chi}\begin{bmatrix} h_1(\chi)\xi^{(1)}_{jm} \\ -ih_2(\chi)\xi^{(2)}_{jm} \end{bmatrix} \qquad (j \geqslant |q| + \tfrac{1}{2}) \tag{2.19}$$

$$\text{Type B} \qquad \psi^{(2)}_{jm} = \frac{1}{\chi}\begin{bmatrix} h_3(\chi)\xi^{(2)}_{jm} \\ -ih_4(\chi)\xi^{(1)}_{jm} \end{bmatrix} \qquad (j \geqslant |q| + \tfrac{1}{2}) \tag{2.20}$$

where

$$\begin{aligned}
&\xi^{(1)}_{jm} = c\phi^{(1)}_{jm} - s\phi^{(2)}_{jm}, \\
&\xi^{(2)}_{jm} = s\phi^{(1)}_{jm} + c\phi^{(2)}_{jm}, \\
&c = q[(2j+1+2q)^{1/2} + (2j+1-2q)^{1/2}]/2|q|(2j+1)^{1/2} \\
&s = q[(2j+1+2q)^{1/2} - (2j+1-2q)^{1/2}]/2|q|(2j+1)^{1/2} \\
&\phi^{(1)}_{jm} = \begin{bmatrix} \left(\dfrac{j+m}{2j}\right)^{1/2} Y_{q,j-1/2,m-1/2} \\ \left(\dfrac{j-m}{2j}\right)^{1/2} Y_{q,j-1/2,m+1/2} \end{bmatrix} \\
&\phi^{(2)}_{jm} = \begin{bmatrix} -\left(\dfrac{j-m+1}{2j+2}\right)^{1/2} Y_{q,j+1/2,m-1/2} \\ \left(\dfrac{j+m+1}{2j+2}\right)^{1/2} Y_{q,j+1/2,m+1/2} \end{bmatrix}.
\end{aligned} \tag{2.21}$$

where $Y_{q,L,M}$ is the monopole harmonic whose basic properties are tabulated in the appendix [6, 27].

In (2.19) and (2.20), $h_i(\chi)$ (i = 1, 2, 3, 4) are defined in a rather different way then in [7], thus, the system of equations satisfied by $h_i(\chi)$ is obtained in the compact form which is easily treated. According to Lemma I of [7], we obtain, for type A:

$$\begin{aligned}
&\left(M - E - \frac{\lambda}{\eta}\right)h_1(\chi) + \partial_\eta h_2(\chi) = -\frac{\mu}{\eta}\frac{\chi}{\sin\chi}h_2(\chi) \\
&\partial_\eta h_1(\chi) + \left(M + E + \frac{\lambda}{\eta}\right)h_2(\chi) = \frac{\mu}{\eta}\frac{\chi}{\sin\chi}h_1(\chi)
\end{aligned} \tag{2.22}$$

and for type B:

$$\begin{aligned}
&\left(M - E - \frac{\lambda}{\eta}\right)h_3(\chi) + \partial_\eta h_4(\chi) = \frac{\mu}{\eta}\frac{\chi}{\sin\chi}h_4(\chi) \\
&\partial_\eta h_3(\chi) + \left(M + E + \frac{\lambda}{\eta}\right)h_4(\chi) = -\frac{\mu}{\eta}\frac{\chi}{\sin\chi}h_3(\chi)
\end{aligned} \tag{2.23}$$

where

$$\mu = [(j + \tfrac{1}{2})^2 - q^2]^{1/2} > 0 \tag{2.24}$$

and $\eta = a\chi$, $q = Zeg$, g is the strength of the magnetic monopole, Dirac quantization is that $eg = n/2$ ($n = 0, \pm 1, \pm 2, \ldots$) [4].

For the flat Robertson–Walker cosmological model, we have, for type A:

$$\begin{aligned} &\left(M - E - \frac{\lambda}{\eta}\right)h_1 + \left(\partial_\eta + \frac{\mu}{\eta}\right)h_2 = 0 \\ &\left(\partial_\eta - \frac{\mu}{\eta}\right)h_1 + \left(M + E + \frac{\lambda}{\eta}\right)h_2 = 0 \end{aligned} \tag{2.25}$$

and for type B:

$$\begin{aligned} &\left(M - E - \frac{\lambda}{\eta}\right)h_3 + \left(\partial_\eta - \frac{\mu}{\eta}\right)h_4 = 0 \\ &\left(\partial_\eta + \frac{\mu}{\eta}\right)h_3 + \left(M + E + \frac{\lambda}{\eta}\right)h_4 = 0 \end{aligned} \tag{2.26}$$

which are the normal Dirac radial equations for the system of a fermion and a Dirac dyon in the flat space-time [11], if η is replaced by r. For the open cosmological model, we have, for type A

$$\begin{aligned} &\left(M - E - \frac{\lambda}{\eta}\right)h_1 + \partial_\eta h_2 = -\frac{\mu}{\eta}\frac{\chi}{\sinh\chi}h_2 \\ &\partial_\eta h_1 + \left(M + E + \frac{\lambda}{\eta}\right)h_2 = \frac{\mu}{\eta}\frac{\chi}{\sinh\chi}h_1 \end{aligned} \tag{2.27}$$

and for type B

$$\begin{aligned} &\left(M - E - \frac{\lambda}{\eta}\right)h_3 + \partial_\eta h_4 = \frac{\mu}{\eta}\frac{\chi}{\sinh\chi}h_4 \\ &\partial_\eta h_3 + \left(M + E + \frac{\lambda}{\eta}\right)h_4 = -\frac{\mu}{\eta}\frac{\chi}{\sinh\chi}h_3. \end{aligned} \tag{2.28}$$

3. Wavefunctions of bound states of a fermion and a Dirac dyon in flat space-time

We can solve (2.25) and (2.26) according to the standard treatment in quantum-mechanics textbooks [29]. When $r \to 0$, equation (2.26) is reduced to

$$\begin{aligned} &(\lambda/\eta)h_1(\eta) - (\partial_\eta + \mu/\eta)h_2(\eta) = 0 \\ &(\partial_\eta - \mu/\eta)h_1(\eta) + (\lambda/\eta)h_2(\eta) = 0. \end{aligned} \tag{3.1}$$

Setting $h_1(\eta) = \alpha\eta^\nu$, $h_2(\eta) = \beta\eta^\nu$, where α and β are constants, from (3.1) we obtain

$$\begin{aligned} &\lambda\alpha - (\nu + \mu)\beta = 0 \\ &(\nu - \mu)\alpha + \lambda\beta = 0. \end{aligned} \tag{3.2}$$

From conditions of non-zero α and β, and the finiteness of $h_1(\eta)$ and $h_2(\eta)$ when $\eta\to 0$, we have

$$\nu=(\mu^2-\lambda^2)^{1/2}=[(j+\tfrac{1}{2})^2-q^2-(ZZ_de^2)^2]^{1/2}>0. \tag{3.3}$$

Setting

$$\rho=2(M^2-E^2)^{1/2}\eta=2p\eta \qquad (p=(M^2-E^2)^{1/2}) \tag{3.4}$$

$$\begin{aligned} h_1(\rho)&=2p(M+E)^{1/2}e^{-\rho/2}\rho^{\nu}\,(Q_1(\rho)+Q_2(\rho))\\ h_2(\rho)&=2p(M-E)^{1/2}e^{-\rho/2}\rho^{\nu}(Q_1(\rho)-Q_2(\rho)) \end{aligned} \tag{3.5}$$

we have

$$\begin{aligned} &\rho Q_1'(\rho)+(\nu-\lambda E/p)Q_1(\rho)-(\mu+\lambda M/p)Q_2(\rho)=0\\ &\rho Q_2'(\rho)+(\nu-\rho-\lambda E/p)Q_2(\rho)-(\mu-\lambda E/p)Q_1(\rho)=0 \end{aligned} \tag{3.6}$$

and

$$\rho Q_1''(\rho)+(2\nu+1-\rho)Q_1'(\rho)-(\nu-\lambda E/p)Q_1(\rho)=0 \tag{3.7}$$

$$\rho Q_2''(\rho)+(2\nu+1-\rho)Q_2'(\rho)-(\nu+1-\lambda E/p)Q_2(\rho)=0. \tag{3.8}$$

If we set

$$\begin{aligned} h_3(\rho)&=2p(M+E)^{1/2}e^{-\rho/2}\rho^{\nu}(Q_3(\rho)-Q_4(\rho))\\ h_4(\rho)&=2p(M-E)^{1/2}e^{-\rho/2}\rho^{\nu}(Q_3(\rho)+Q_4(\rho)) \end{aligned} \tag{3.9}$$

then $Q_3(\rho)$ satisfies (3.7), and $Q_4(\rho)$ satisfies (3.8).

Equations (3.7) and (3.8) are standard confluent hypergeometric equations. Their finite solutions at the origin are the confluent hypergeometric function $F(\alpha, \beta, \rho)$

$$\begin{aligned} Q_{1,3}(\rho)&=A_{1,3}F(\nu-\lambda E/p, 2\nu+1, \rho)\\ Q_{2,4}(\rho)&=A_{2,4}F(\nu+1-\lambda E/p, 2\nu+1, \rho) \end{aligned} \tag{3.10}$$

from (2.25), (3.5) and (3.10), the radial wavefunctions are

$$\begin{aligned} R_1(\rho)\equiv 2ph_1(\rho)/\rho=4p^2(M+E)^{1/2}e^{-\rho/2}\rho^{\nu-1}[&A_1F(\nu-\lambda E/p, 2\nu+1, \rho)\\ &+A_2F(\nu+1-\lambda E/p, 2\nu+1, \rho)]\\ R_2(\rho)\equiv 2ph_2(\rho)/\rho=4p^2(M-E)^{1/2}e^{-\rho/2}\rho^{\nu-1}[&A_1F(\nu-\lambda E/p, 2\nu+1, \rho)\\ &-A_2F(\nu+1-\lambda E/p, 2\nu+1, \rho)]. \end{aligned} \tag{3.11}$$

When $\rho\to 0$, $F(\alpha, \beta, \rho)\to 0$, from (3.6), we have

$$A_2=\frac{\nu-\lambda E/p}{\mu+\lambda E/p}A_1. \tag{3.12}$$

Similarly, for A_3 and A_4, we have

$$A_4=\frac{\nu-\lambda E/p}{\mu-\lambda M/p}A_3. \tag{3.13}$$

When $\rho\to\infty$, $F(\alpha,\beta,\rho)\to e^{\rho}$, so $R_i(\rho)$ is divergent. In order to avoid the divergence, we must set $\nu-\lambda E/p=-n$, $(n=0, 1, 2, \ldots)$ and $\nu+1-\lambda E/p=-m$, $(m=0, 1, 2, \ldots)$. When $n=0$, $F(\nu-\lambda E/p,\ 2\nu+1,\ \rho)$ is finite, but $F(\nu+1-\lambda E/p,$

$2\nu+1,\rho)$ is still divergent. In order to make $R_i(\rho)$ finite, we must have $n=m+1$, $(m=0, 1, 2, \ldots)$, so

$$\nu-\lambda E/p=-n, (n=1,2,3,\ldots). \tag{3.14}$$

Thus we obtain

$$\begin{aligned} E_{n,q,j} &= \pm M\left[1+\frac{1}{(n+\nu)^2/\lambda^2}\right]^{-1/2} \\ &= \pm M\left(1+\left[\frac{ZZ_{\rm d}e^2}{n+[(j+\frac{1}{2})^2-q^2-(ZZ_{\rm d}e^2)^2]^{1/2}}\right]^2\right)^{-1/2} \end{aligned} \tag{3.15}$$

where $n=1, 2, 3, \ldots$; $j\geqslant|q|+\frac{1}{2}$; $q=Zeg\neq0$; $eg=\pm\frac{1}{2}, \pm1, \pm\frac{3}{2}, \ldots$; $\mu=[(j+\frac{1}{2})^2-q^2]^{1/2}>0$; $\lambda=ZZ_{\rm d}e^2$. Notice that the total angular momentum j, which is defined in [7], is different from the total angular momentum in ordinary quantum mechanics. Here, j can take integer as well as half-integer values. The spectrum (3.15) is hydrogen-like, but is different from the atomic or molecular spectrum.

By using the relation

$$L_n^\nu(x)=\frac{\Gamma(\nu+1+n)}{n!\Gamma(\nu+1)}F(-n,\nu+1,x) \tag{3.16}$$

and (3.12) and (3.14), (3.11) is reduced to

$$\begin{aligned} R_{1,n,q,j} &= 4p_{n,q,j}^2(M+E_{n,q,j})^{1/2}A_{1n,q,j}e^{-\rho/2}\rho^{\nu-1}\left\{\frac{n!\Gamma(2\nu+1)}{\Gamma(2\nu+1+n)}L_n^{2\nu}(\rho)\right. \\ &\quad\left.-\frac{n}{\mu+(\mu^2+n^2+2n\nu)^{1/2}}\frac{(n-1)!\Gamma(2\nu+1)}{\Gamma(2\nu+n)}L_{n-1}^{2\nu}(\rho)\right\} \\ R_{2,n,q,j} &= 4p_{n,q,j}^2(M-E_{n,q,j})^{1/2}A_{1n,q,j}e^{-\rho/2}\rho^{\nu-1}\left\{\frac{n!\Gamma(2\nu+1)}{\Gamma(2\nu+1+n)}L_n^{2\nu}(\rho)\right. \\ &\quad\left.+\frac{n}{\mu+(\mu^2+n^2+2n\nu)^{1/2}}\frac{(n-1)!\Gamma(2\nu+1)}{\Gamma(2\nu+n)}L_{n-1}^{2\nu}(\rho)\right\}. \end{aligned} \tag{3.17}$$

Similarly, we have

$$\begin{aligned} R_{3,n,q,j} &= 4p_{n,q,j}^2(M+E_{n,q,j})^{1/2}A_{3n,q,j}e^{-\rho/2}\rho^{\nu-1}\left\{\frac{n!\Gamma(2\nu+1)}{\Gamma(2\nu+1+n)}L_n^{2\nu}(\rho)\right. \\ &\quad\left.+\frac{n}{\mu-(\mu^2+n^2+2n\nu)^{1/2}}\frac{(n-1)!\Gamma(2\nu+1)}{\Gamma(2\nu+n)}L_{n-1}^{2\nu}(\rho)\right\} \\ R_{4,n,q,j} &= 4p_{n,q,j}^2(M-E_{n,q,j})^{1/2}A_{3n,q,j}e^{-\rho/2}\rho^{\nu-1}\left\{\frac{n!\Gamma(2\nu+1)}{\Gamma(2\nu+1+n)}L_n^{2\nu}(\rho)\right. \\ &\quad\left.-\frac{n}{\mu-(\mu^2+n^2+2n\nu)^{1/2}}\frac{(n-1)!\Gamma(2\nu+1)}{\Gamma(2\nu+n)}L_{n-1}^{2\nu}(\rho)\right\}. \end{aligned} \tag{3.18}$$

Because $\xi^{(1)}_{jm}$ and $\xi^{(2)}_{jm}$ are normalized, radial wavefunctions satisfy the following normalization condition

$$\int_0^\infty \sum_{i=1(3)}^{2(4)} |R_i(2p_{n,q,j}\chi)|^2 \chi^2 \mathrm{d}\chi = 1. \tag{3.19}$$

Using the normalization condition of $L_n^\nu(z)$

$$\int_0^\infty \mathrm{d}x\, x^\nu \mathrm{e}^{-x} L_n^\nu(x) L_{n'}^\nu(x) = \frac{\Gamma(\nu+n+1)}{n!}\delta_{nn'} \tag{3.20}$$

we have

$$A_{1,n,q,j} = \frac{1}{2\Gamma(2\nu+1)}\left(MP_{n,q,j}\frac{n!}{\Gamma(2\nu+n)}\left[\frac{1}{2\nu+n} + \frac{n}{[\mu+(\mu^2+n^2+2\nu n)^{1/2}]^2}\right]\right)^{-1/2}$$
$$A_{3,n,q,j} = \frac{1}{2\Gamma(2\nu+1)}\left(Mp_{n,q,j}\frac{n!}{\Gamma(2\nu+n)}\left[\frac{1}{2\nu+n} + \frac{n}{[\mu-(\mu^2+n^2+2\nu n)^{1/2}]^2}\right]\right)^{-1/2}. \tag{3.21}$$

4. The energy levels of the dyon–fermion system in closed and open cosmological models

It is difficult to find an exact solution in closed or open cosmologial model. We estimate the influence of terms including $\chi/\sin\chi$ and $\chi/\sinh\chi$ on the energy levels, by expanding them and considering the first approximation

$$\frac{\chi}{\sin\chi} = 1 + \frac{1}{6}\left(\frac{\eta}{a}\right)^2$$
$$\frac{\chi}{\sinh\chi} = 1 - \frac{1}{6}\left(\frac{\eta}{a}\right)^2. \tag{4.1}$$

Since the region of the system is about of the order of magnitude of the Bohr radius, which is very much less than the cosmological radius, $\pm\frac{1}{6}(\eta/a)^2$ can be taken as a perturbation. We know the energy levels $E_{n,q,j}$ of the dyon–fermion system in the flat space-time. We also can obtain the energy levels of the closed and open cosmological models by the method of perturbation expansion

$$E_{n,q,j} = \pm M\left(1 + \left[\frac{ZZ_\mathrm{d}e^2}{n+[\mu^2-(ZZ_\mathrm{d}e^2)^2]^{1/2}}\right]^2\right)^{-1/2}$$
$$\times\left(1 \pm \frac{1}{6}\frac{\mu^2(ZZ_\mathrm{d}e^2)^2(A_b/a)^2}{[\mu^2-(ZZ_\mathrm{d}e^2)^2]^{1/2}(n+[\mu^2-(ZZ_\mathrm{d}e^2)^2]^{1/2})(n^2+\mu^2+2n[\mu^2-(ZZ_\mathrm{d}e^2)^2]^{1/2})}\right) \tag{4.2}$$

where A_b is the average radius of region of the fermion–dyon system and there is a positive sign for the closed space-time and a negative sign for the open space-time; $n=1,2,3,\ldots$; $j \geqslant |q| + \frac{1}{2}$; $q = Zeg \neq 0$, $eg = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots$; $\mu = [(j+\frac{1}{2})^2 - q^2]^{1/2} > 0$. Notice that the total angular momentum j, which is defined in [7], is different from the total angular momentum in ordinary quantum mechanics. Here, j can take integer as well as half-integer values. It is worth mentioning that (2.22) and (2.23) in the closed Robertson–Walker metric and (2.27) and (2.28) in the open Robertson–Walker metric return to the normal Dirac equation, when $a \to \infty$. Similarly the energy levels (4.2) also return to normal energy levels (3.15), when $a \to \infty$. Therefore, the Dirac equation of the fermion–dyon system has been extended to the case of the Robertson–Walker metric in terms of a spin coefficient method. The energy spectrum (4.2) is hydrogen-like but it is quite different from the ordinary hydrogen-like one. If dyons exist in the universe, (4.2) leads to the possibility of looking for dyonic bound states, for example, from astronomical observations. Let us discuss the energy spectrum (4.2) as follows:

(i) When q takes half-integer values, the total angular momentum j of this system takes integer values, leading to a new series of energy spectra that do not exist in the ordinary hydrogenlike atom.

(ii) In the case of flat space-time, when q takes integer values, j takes half-integer values, as with the case of the ordinary hydrogen-like atom. But compared with the energy level of the hydrogen-like atom

$$E^H_{n,j} = M\left[1 + \frac{(Ze^2)^2}{(n + [(j+\frac{1}{2})^2 - (Ze^2)^2]^{1/2})^2}\right]^{-1/2} \tag{4.3}$$

$E_{n,q,j}$ is shifted down. Now we consider the amount of shift. Let the dyon charge $Z_d = +1$. Take $Z = -1$, $|q| = 1$, $j = \frac{3}{2}$. For the $n = 1$ energy level, from (4.2) and (4.3), we have $(E_{q,n,j} - E^H_{n,j})/M \approx 10^{-2}\alpha^2$. On the other hand, comparison of $\Delta E = E(n'=1) - E(n=0)$, (4.2) and (4.3) shows $[\Delta E_{q,n,j} - \Delta E^H_{n,j})/M \approx 10^{-2}\alpha^2$. Let it be noted that these differences can be measured by existing experimental techniques.

(iii) The search for free monopoles has not yielded any definite results due to many difficulties. First, we do not know how small the monopole flux is (for example, according to Parker's limit $\Phi < 3 \times 10^{-9}\,\mathrm{cm}^{-2}\,\mathrm{yr}^{-1}$ [30]), so we do not know how long we have to wait before we record a possible event; second, the estimation of monopole mass is model-dependent, for example, the mass of the classical monopole is about the order of 10–10^2 GeV, and the mass of the superheavy monopole in grand unified theories is about the order of 10^{16} GeV, but we are ignorant of its definite value. This brings about difficulty in searching for a monopole in accelerator experiments (if the mass of the monopole is within the energy region reached by accelerators). On the other hand, the approach to the search for dyons (or monopoles) under the bound condition, compared with the approach to the search for the free monopole, may be somewhat easier to perform. Because (a) the superheavy dyon is treated as an external potential so that its mass does not appear in the energy spectrum formula (4.2); (b) if dyons were plentifully produced in the early universe and formed into bound states with charged fermions, perhaps the radial electromagnetic spectra of the bound systems have already been recorded on astronomical observations over a long period.

(iv) After the epoch of recombination, the cosmological radius $a(\tau)$ is about $5 \times 10^5\,\mathrm{ly} \approx 5 \times 10^{23}$ cm, so $\frac{1}{6}(A_b/a)^2 \approx 10^{-63}$. This is too small to be observed even by modern techniques. Z_d is the electric charge of the dyon which need not be an integer. If we consider that the dyon has a large charge Z_d, the revision term of (4.2) would become much larger and more important, when ZZ_de^2 is near $\mu = [(j+\frac{1}{2})^2 - q^2]^{1/2}$.

(v) Of course, any attempt to detect monopoles or dyons is a challenging enterprise because if they still exist in the present universe they are surely rare.

In conclusion, the energy spectrum (4.2) may initiate a promising new approach to the search for dyons in the bound condition. If dyons are indeed present in the universe, they have an interesting astronomical observation.

Acknowledgments

XZL would like to thank professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. This work was supported in part by the National Natural Science Foundation of China under Grant No. 19175016 (XZL) and 1880126 (JZZ).

Appendix 1. Some properties of monopole harmonics [6, 27]

$$\hat{L}^2 Y_{q,L,M} = L(L+1) Y_{q,L,M},$$
$$\hat{L}_Z Y_{q,L,M} = M Y_{q,L,M}$$
$$L = |q|, |q|+1, |q|+2, \ldots, \qquad M = -L, -L+1, \ldots, L. \tag{A.1}$$

$$\int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi Y^*_{q,L',M'}(\theta\cdot\varphi) Y_{q,L,M}(\theta,\varphi) = \delta_{L'L}\delta_{M'M}. \tag{A.2}$$

For the fixed q, $Y_{q,L,M}$ is orthogonal and normalized.

$$Y_{0,L,M}(\theta,\varphi) = Y_{L,M}(\theta,\varphi) \tag{A.3}$$

which is ordinary harmonics.

$$Y_{q,L,M} Y_{q',L',M'} = \sum_{L''} (-1)^{L+L'+L''+q''+M''} \left[\frac{(2L+1)(2L'+1)(2L''+1)}{4\pi}\right]^{1/2} \times \begin{pmatrix} LL'L'' \\ MM'M'' \end{pmatrix} \begin{pmatrix} LL'L'' \\ qq'q'' \end{pmatrix} Y_{-q'',L'',-M''}. \tag{A.4}$$

where $M'' = -M - M'$, $q'' = -q - q'$, L'' takes all the possible values of coupled L and L'. (A.4) is the addition theorem of monopole harmonics.

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